[This question paper contains 8 printed pages.]

Your Roll No..............

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Sr. No. of Question Paper : 1441

Unique Paper Code : 2352571201

Name of the Paper : Elementary Linear Algebra

Name of the Course : **B.A. (Prog.)**

Semester : II – DSC

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** question by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator not allowed.

P.T.O.

1. (a) If x and y are vectors in R^n, then prove that $||x + y|| \le ||x|| + ||y||$.

Also verify it for the vectors x = [−1, 4, 2, 0, −3] and y = [2, 1, −4, −1, 0] in R^5. (5.5+2)

(b) Prove that for vectors x and y in R^n,

(i) $x . y = \frac{1}{4}(||x + y||^2 - ||x - y||^2)$

(ii) If $(x + y) . (x - y) = 0$, then $||x|| = ||y||$. (4+3.5)

(c) Solve the systems $AX = B_1$ and $AX = B_2$ simultaneously, where

$$A = \begin{bmatrix} 9 & 2 & 2 \\ 3 & 2 & 4 \\ 27 & 12 & 22 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -6 \\ 0 \\ 12 \end{bmatrix}, \text{ and } \quad B_2 = \begin{bmatrix} -12 \\ -3 \\ 8 \end{bmatrix}$$

(7.5)

2. (a) Find the reduced row echelon form of the following matrix :

$$A = \begin{bmatrix} 2 & -5 & -20 \\ 0 & 2 & 7 \\ 1 & -5 & -19 \end{bmatrix}$$ (7.5)

(b) Express the vector x = [2, −1, 4] as a linear combination of vectors v_1 = [3, 6, 2] and v_2 = [2, 10, −4], if possible. (7.5)

(c) Define the rank of a matrix and determine it for the following matrix :

$$B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{bmatrix}$$ (1.5+6)

3. (a) Check if the following matrix is diagonalizable or not :

$$\begin{bmatrix} 3 & 4 & 12 \\ 4 & -12 & 3 \\ 12 & 3 & -4 \end{bmatrix}$$ (7.5)

(b) Show that the set of all polynomials P(x) forms a vector space under usual polynomial addition and scalar multiplication. (7.5)

(c) Give an example of a finite dimensional vector space. Check if the following are a vector space or not :

(i) R^2 with the addition $[x, y] \oplus [w, z] = [x + w + 1, y + z - 1]$ and scalar multiplication $a \otimes [x, y] = [ax + a - 1, ay - 2]$.

(ii) set of all real valued functions $f : R \to R$ such that $f\left(\frac{1}{2}\right)=1$, under usual function addition and scalar multiplication.

(1.5+3+3)

4. (a) Definesubspace of a vector space. Further show that intersection of two subspaces of a vector space V is a subspace of V. (1.5+6)

(b) Define a linearly independent set. Check if $S = \{(1, -1, 0, 2), (0, -2, 1, 0), (2, 0, -1, 1)\}$ is linearly independent set in R^4 or not.

(1.5+6)

(c) Define an infinite dimensional and finite dimensional vector space.

Consider the set of all real polynomials denoted by $P(x)$, and the set of all real polynomials of degree at most n denoted by $P_n(x)$. Describe a basis of $P(x)$ and $P_n(x)$ and mention if these are finite dimensional or infinite dimensional.

(2+4+1.5)

5. (a) Show that the mapping $L : M_{nn} \rightarrow M_{nn}$, defined as $L(A) = A + A^T$ is a linear operator, where M_{nn} is set of $n \times n$ matrices and A^T denotes the transpose of the matrix A. Find the Kernel of L.

(3+4.5)

(b) Let $L: R^2 \rightarrow R^3$ be a linear transformation defined as $L\{[a, b]) = [a - b, a, 2a + b]$. Find the matrix of linear transformation A_{BC} of L, with respect to the basis $B = \{[1,2], [1,0]\}$ and $C = \{[1,1,0], [0,1,1], [1,0,1]\}$. (7.5)

2. (c) Let $L: V \rightarrow W$, be a linear transformation, then define Ker(L), Range(L). Further show that Ker(L) is a subspace of V and Range(L) is a subspace of W. (1.5+1.5+2.5+2)

6. (a) For the linear transformation $L: R^3 \rightarrow R^3$ defined as

$$L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Find Ker(L) and Range(L). (4+3.5)

(b) Let $L: V \rightarrow W$ be a one-to-one linear transformation. Show that if T is a linearly independent subset of V, then L(T) is a linearly independent subset of W. (7.5)

(c) For the linear transformation L: $R^2 \rightarrow R^2$, defined as :

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Find L^{-1}, if it exists. (7.5)

2024 June

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Your Roll No..............

Sr. No. of Question Paper : 1242 **F**

Unique Paper Code : 2352011203

Name of the Paper : Ordinary Differential Equations

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester / Type : II / DSC

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts of each question.
3. Each part carries **7.5** marks.
4. Use of non-programmable Scientific Calculator is allowed.

1. (a) Solve the initial value problem

$$(e^{2x}y^2 - 2x)\,dx + e^{2x}y\,dy = 0,\ y(0) = 2$$

(b) Solve

$$(2x + \tan y)\, dx + (x - x^2 \tan y)\, dy = 0$$

(c) Solve

(i) $(3x^2 + 4xy - 6)\, dy + (6xy + 2y^2 - 5)\, dx = 0$

(ii) $\frac{d^2y}{dx^2} = 2y\left(\frac{dy}{dx}\right)^3 = 2y$ by reducing the order.

2. (a) A certain rumor began to spread one day in a city with a population of 100,000. Within a week, 10,000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population has heard the rumor?

(b) The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident in a certain region has left the level of cobalt to be 100 times the acceptable level for habitation. How long will it be until the region is again habitable?

(c) A cake is removed from an oven at 210°F and left to cool at room temperature of 70°F. After 30 minutes, the temperature of the cake is 140°F. What will be its temperature after 40 minutes? When will the temperature be 100°F?

3. (a) Show that the solutions x, x^2, x log x of the third order differential equation

$$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$$

are linearly independent on $(0, \infty)$. Also find the particular solution satisfying the given initial condition.

$$y(1) = 3,\ y'(1) = 2,\ y''(1) = 1$$

(b) Solve the differential equation using the method of Variation of Parameters

$$\frac{d^2y}{dx^2} + 9y = \tan 3x$$

(c) Find the general solution of the differential equation using the method of undetermined Coefficients.

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}=4e^{-x}+3x^2$$

4. (a) Use the operator method to find the general solution of the following linear system

$$2\frac{dx}{dt}+\frac{dy}{dt}+x+5y=4t$$

$$\frac{dx}{dt}+\frac{dy}{dt}+2x+2y=2$$

(b) Solve the initial value problem. Assume $x > 0$.

$$x^2\frac{d^2y}{dx^2}-5x\frac{dy}{dx}+8y=2x^3,\quad y(2)=0,\ y'(2)=8$$

(c) A body with mass $m = \frac{1}{2}$ kg is attached to the end of a spring that is stretched 2m by a force of 16N. It is set in motion with initial position $x_0 = 1m$ and initial velocity $v_0 = -5m/s$. Find the position function of the body as well as the amplitude, frequency and period of oscillation.

5. (a) Define the term Carrying Capacity. Derive the logistic equation

$$\frac{dX}{dt}=rX\left(1-\frac{X}{K}\right)$$

where K is the carrying capacity of the population. Also find the solution.

(b) The per-capita death rate for the fish is 0.5 fish per day per fish, and the per-capita birth rate is 1.0 fish per day per fish. Write a word equation describing the rate of change of the fish population. Hence obtain a differential equation for the number of fish. If the fish population at a given time is 240, 000, give an estimate of the number of fish born in one week.

(c) In an epidemic model where infected get recovered, the differential equation is of the form

$$\frac{dS}{dt} = -\beta SI, \ \frac{dI}{dt} = \beta SI - \gamma I$$

Use parameter values $\beta = 0.002$ and $\gamma = 0.4$, and assume that initially there is only one infective but there are 500 susceptibles. How many susceptibles never get infected, and what is the maximum number of infectives at any time? What happens as time progresses, if the initial number of susceptibles is doubled, $S(0) = 1000$? How many people were infected in total.

6. (a) A public bar opens at 6 p.m. and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators that exchange the smoke-air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20m by 15m, and a height of 4m. It is estimated that smoke enters the room at a constant

rate of $0.006 \text{ m}^3/\text{min}$, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced. The problem is to establish a good time to leave the bar, that is, sometime before the concentration of carbon monoxide reaches the lethal limit. Starting from a word equation or a compartmental diagram, formulate the differential equation for the changing concentration of carbon monoxide in the bar over time. By solving the equation above, establish at what time the lethal limit will be reached.

(b) Find the equilibrium solution of the differential equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)$$

And discuss the stability of equilibrium solution.

(c) Consider a disease where the infected get recovered. A model describing this is given by the differential equations

$$\frac{dS}{dt} = -\beta SI, \ \frac{dI}{dt} = \beta SI - \gamma I$$

Use chain rule to find arelation between S and I. Obtain and sketch the phase-plane curves. Determine the direction of travel along the trajectories.